Prep Camp

Sets and Functions

Gabor Wiese

Lecture 1

Sets

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- how to make new sets out of given sets.

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Exercise 1.

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There is a notion of cardinality for infinite sets, showing that $\mathbb{R}$ is strictly larger than $\mathbb{Q}$, but that $\mathbb{Q}$ and $\mathbb{N}$ have the same cardinality.

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Let $n \in \mathbb{N}$ and $A_1, A_2, \dots, A_n$ be sets.
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Example: Let $A := \{1, 2\}$ and $B := \{a, b, c\}$.

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Theorem. If $\#A = n$, then $\#\mathcal{P}(A) = 2^n$.

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Exercise 8.

SOME PROPERTIES (I)

Lemma. Let A, B, C be sets. Then:

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Some properties (I)

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Lemma. Let A, B, C be assertions. Then:

1. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$
2. $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$

We prove:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
for sets A, B, C, using the corresponding rule from logic.

ONE PROOF

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We have thus proved that an object x is in $A \cap (B \cup C)$ if and only if it is in $(A \cap B) \cup (A \cap C)$. Hence, the two sets are equal.

Lemma. Let S be a set and let $A \subseteq S$ and $B \subseteq S$ be subsets. Then:

1. $A \sqcup (S \setminus A) = S$

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Homework: do as many of the remaining exercises as possible.

Lecture 2

Functions

EXAMPLE

Let us describe the function $f(x) = x^2$.

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Instead of $x \mapsto x^2$, can also write the rule: $f(x) = x^2$.

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When defining a function/map, always specify domain and value set!

Definition. Let A, B be sets. A function/map

$$f : A \to B$$

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- $g : \{X \mid X \text{ is a student in PrepCamp}\} \to \{male, female\}$
  $X \mapsto$ gender of X.

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Consider the following keeping in mind the rules:

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Exercise 1.

Important Examples

- Let A be a set. The identity map on A is:
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- Let A be a set. The identity map on A is:
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- Let $f : A \to B$ be a map and $S \subseteq A$ a subset. The restriction of f to S is the map
  $f|_S : S \to B, \quad s \mapsto f(s).$

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Exercise 2.

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Exercise 3.

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Exercise 4.

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4. Suppose $\#A = \#B$. Then
   f is injective $\Leftrightarrow$ f is surjective $\Leftrightarrow$ f is bijective

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Proof.

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INVERSE MAPS

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- (A) Suppose $f : A \to B$ is bijective. Then f possesses a unique inverse $f^{-1} : B \to A$.
  Namely, for $b \in B$, by the bijectivity of f, there is a unique a such that $f(a) = b$; define $f^{-1}(b) := a$.
- (B) Suppose that $f : A \to B$ possesses an inverse $g : B \to A$. Then f is bijective.

Proof.
Since $f \circ g = \mathrm{id}_B$ holds, f is surjective.
Since $g \circ f = \mathrm{id}_A$ holds, f is injective.

Exercise 6.