Geometry of the $p-$adic upper half plane

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1 Basic notations

We use the following notation

- K denote a finite extension of the $p-$adic numbers $\mathbf{Q}_p$.
- G be the group $GL_2(K)$.
- $\mathcal{O}_K$ denotes the ring of integers in K.
- π is an uniformizing parameter for $\mathcal{O}_K$.
- $|\cdot|$ is the normalized $p-$adic absolute value on K extending the $p-$adic absolute value on $\mathbf{Q}_p$.
- $\omega : K \to \mathbb{Z} \cup \{\infty\}$ is the additive valuation normalized so that $\omega(\pi) = 1$.

Let V be a fixed two dimensional vector space over K, viewed as a space of row vectors, on which G acts on the left by the formula

$$g([x, y]) = [x, y] \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$\mathbb{P}^1$ will be $\mathbb{P}(V)$ with its $G-$action. Let V^* be the dual of V , consider e_0 and e_1 elements in V^* respect to the standard basis vectors $[1, 0]$ and $[0, 1]$ in V; they are homogeneous coordinates on $\mathbb{P}^1$. A linear form in e_0 and e_1 is called unimodular if at least one of its two coefficients is a unit in $\mathcal{O}_K$.

The coordinate function

$$z = \frac{e_0}{e_1}$$

is acted on by a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ through the formula

$$\begin{aligned} g_*(z)([x,y]) &= z(g^{-1}([x,y])) \\ &= z([x,y]g) \\ &= z([ax+cy, bx+dy]) \\ &= \frac{az+c}{bz+d} \end{aligned}$$

2 The p-adic upper half plane

We want to study here the p-adic upper half plane $\mathcal{X}$, an space whose L points are given by the rule

$$\mathcal{X}(L) = \mathbb{P}^1(L) \setminus \mathbb{P}^1(K)$$

for complete extensions fields L of K.

Definition 2.1. *A connected affinoid subset of $\mathbb{P}^1$ is the complement of any non-empty finite union of open disks. An affinoid subset of $\mathbb{P}^1$ is a finite union of connected affinoid subsets.*

Definition 2.2. *Given $x \in \mathbb{P}^1(\mathbb{C}_p)$ we may choose homogeneous coordinates $[x_0, x_1]$ for x that are unimodular meaning that both coordinates are integral, but at least one is not divisible by π. For a real number $r > 0$, let*

$$B^-(x,r) = \{y \in \mathbb{P}^1(\mathbb{C}_p) : \omega(y_0x_1 - y_1x_0) > r\}$$

where we always take a unimodular representative $[y_0, y_1]$ of y.

Definition 2.3. *For each integer $n > 0$, let $\mathcal{P}_n$ be a set of representatives for the points of $\mathbb{P}^1(K)$ modulo π^n. Let $\mathcal{X}_n^-$ be the set*

$$\mathcal{X}_n^- := \mathbb{P}^1(C_p) \setminus \bigcup_{x \in \mathbb{P}_n} B^-(x, n-1)$$

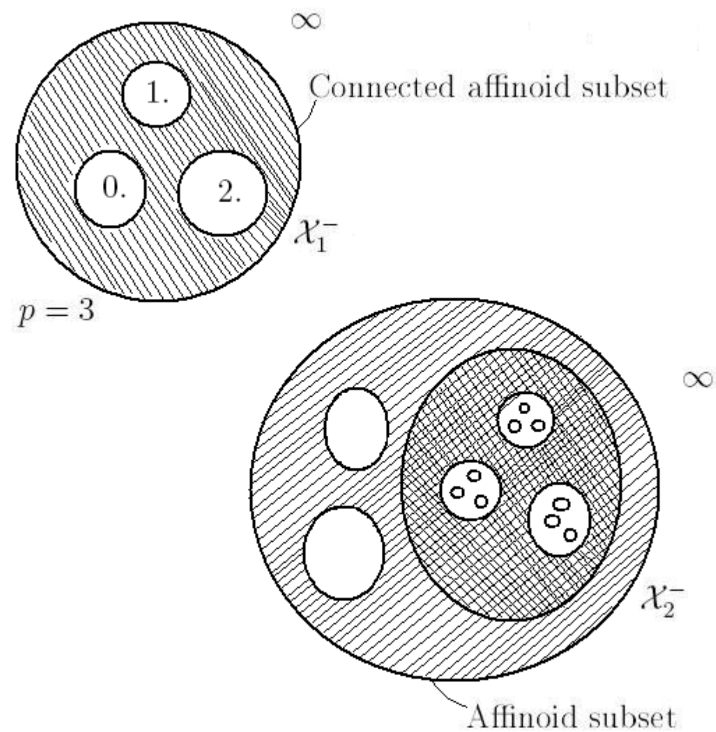

Figure 1: Example

Essencially $\mathcal{X}_n^-$ is constructed by deleting from $\mathbb{P}^1$ smaller and smaller balls around the rationals points.

Definition 2.4. *$\Omega \subseteq \mathbb{P}^1$ is an admissible subset if only if there exists an affinoid covering $\{U_i\}_{i\in I}$ such that, for all affinoid $U \subset \Omega$ there exists $I_n \subset I$ finite such that $U \subseteq \bigcup_{i\in I_n} U_i$. Such affinoid covering is called an admissible covering.*

Proposition 2.5. *$\mathcal{X} = \bigcup_n \mathcal{X}_n^-$ is an admissible open subdomain of $\mathbb{P}^1$ and the coverings by the familie $\{\mathcal{X}_n^-\}_{n=1}^{\infty}$ are admissible coverings.*

2.1 The ring $\mathcal{O}_{\mathcal{X}}$ of entire functions on $\mathcal{X}$

Consider the set

$$\mathcal{O}_{\mathcal{X}_n^-} = \left\{ f : \mathcal{X}_n^- \to \mathbb{C}_p : \text{such that } \exists f_m \to f, f_m \text{ is rational with poles outside of } \mathcal{X}_n^- \right\}$$

$$\mathcal{O}_{\mathcal{X}} = \left\{ f : \mathcal{X} \to \mathbb{C}_p : \text{such that } f/\mathcal{X}_n^- \in \mathcal{O}_{\mathcal{X}_n^-} \text{ for all } n \right\}$$

Remark 2.6. Here the norm for the convergence is the norm of the suprem.

Proposition 2.7. *$\mathcal{O}_{\mathcal{X}}$ is a Fréchet space with this norm.*

3 The Reduction Map

3.1 The Bruhat-Tits Tree

Now consider the fixed two dimensional vector V^* over K.

Definition 3.1. *A lattice L in V^* is a free rank two $\mathcal{O}_k$ module in V^*. We define the following equivalence relation on the set of lattices in V^*, $L_1 \sim L_2$ if there is an scalar $a \in K$ such as $L_1 = aL_2$.*

Definition 3.2. *Let X be the graph whose vertices are equivalence classes $[L]$ of lattices $L \subset V^*$, where two vertices x and y are joined by an edge if $x = [L_1]$ and $y = [L_2]$ with*

$$\pi L_1 \subsetneq L_2 \subsetneq L_1.$$

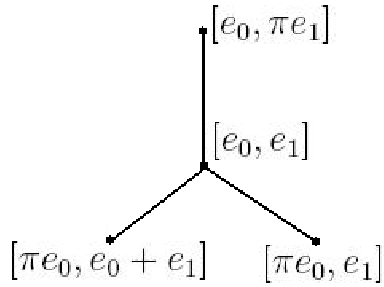

Figure 2: Example

Proposition 3.3. *The graph X is a homogeneous tree of degree $q+1$.*

Remark 3.4. The degree $q+1$ means that in every vertice there are exactly $q+1$ edges leaving and for a tree we means that the graph is connected and have no loops.

Proof. **Degree $q+1$:** Let $[L_1]$ a class of equivalence of $[L_2]$ is another vertex leaving $[L_1]$ then

$$\pi L_1 \subsetneq L_2 \subsetneq L_1 \text{ so } \{0\} \subsetneq L_2/\pi L_1 \subsetneq L_1/\pi L_1 \approx (\mathbb{F}_q)^2$$

since $dim\,(L_2/\pi L_1) = 1$ it correspond to the one-dimensional subspaces in $\mathbb{F}_q^2$ and there are exactly $q+1$, so the degree of X is $q+1$.

Connected: Let $[L]$ and $[L']$ two vertices suppose that $L' \subsetneq L$ then a *Jordan-Hölder* sequences for L/L' gives a sequence of lattices

$$L' = L_n \subsetneq L_{n-1} \subsetneq \ldots \subsetneq L_0 = L$$

such that $l(L_{i-1}/L_i) = 1$ for $1 \leq i \leq n$ and the classes $[L_0], [L_1], \ldots, [L_n]$ define a path between $[L]$ and $[L']$.

X **have no loops:** Now suppose that X is not a tree, then a cycle in X should be represented by a chain of lattices

$$L_{d+1} = L' \subsetneq L_d \subsetneq L_{d-1} \subsetneq \ldots \subsetneq L_1 \subsetneq L_0 = L$$

minimal with no equivalent lattices, where $L' = \pi^r L$.

Considering the exact sequences

$$0 \to L_i/L_{i+1} \to L/L_{i+1} \to L/L_i \to 0$$

if and fact that L/L' is not a cyclic $\mathcal{O}_k-$ module we can prove that there is i_0 such that

$$i_0 = \min\{i : L/L_i \text{ is cyclic but } L/L_{i+1} \text{ is not}\}$$

so L_{i_0-1}/L_{i_0+1} is a non cyclic length *two* $-\ \mathcal{O}_k$ module and finally $L_{i_0+1} = \pi L_{i_0-1}$, which is a contradiction. □

In this way we have constructed a combinational object X. For a point $x \in X$ on the vertex determined by $[L]$ and $[L']$ we can write $x = (1-t)[L] + t[L']$; to indicate that the point is at distance t from $[L]$ in direction $[L']$. In this way we can see each edge of X as a copy of $[0,1]$ and we obtain a topological space called **the realization of X**.

3.2 Norms

Definition 3.5. *A norm on V^* is a function $\gamma : V^* \to \mathbb{R} \cup \{\infty\}$ such that*

- $\gamma(x) = \infty$ *if and only if* $x = 0$
- $\gamma(ax) = \omega(a) + \gamma(x)$ *for all* $a \in K$
- $\gamma(x+y) \geq \inf\{\gamma(x), \gamma(y)\}$

We say that $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 - \gamma_2 = c$ for some $c \in \mathbb{R}$.
Now to a point $x \in X$ we associate an equivalence class of norms of V^*. Here we consider two cases:
Case 1 x is a vertex, in this case choose a lattice $L = \langle l_0, l_1 \rangle$ and let

$$\gamma(al_0 + bl_1) = \inf\{\omega(a), \omega(b)\}$$

or alternatively

$$\gamma(w) = -\inf\{n \in \mathbb{Z} : \pi^n w \in L\}$$

Case 2 x lies on a edge, then $x = (1-t)[L] + t[L']$ in this case choose $L = \langle l_0, l_1 \rangle$ and by the Theorem of the principal divisors we can choose $L' = \langle l_0, \pi l_1 \rangle$ and define

$$\gamma(al_0 + bl_1) = \inf\{\omega(a), \omega(b) - t\}$$

Proposition 3.6. *This construction establishes a bijection between the set of equivalence classes of norms on V^* and the points of the space X.*

Proof. We will construct the inverse map of the construction given.
Let γ be any norm on V^*, suppose that $\exists x \in V^*$ such that $\gamma(x) = 0$ (we can scale γ by translating it in its equivalence class). Choose a (finite) set of representatives R in L for the projective space $P(L'/\pi L')$. The norm is determined by its values on elements of R, all of which lie in $[0, 1)$.
Let $w \in V^*$ such that $w = u\pi^m r + \pi^{m+1} w'$ with $u \in \mathcal{O}_K^*$ and $w' \in L'$. Then $\gamma(w) = m + \gamma(r)$, if $\gamma(r) = 0$ for all $r \in R$ then this norm comes from the case 1.
We can check also that norms equivalent to γ have unit balls equivalent to L', if $\gamma(r) > 0$ the $\gamma(r)$ is unique because if $\exists r' \in R$ such that $\gamma(r') > 0$ then $L' = \langle r, r' \rangle$ and $\gamma(x) > 0$ for all $x \in L'$, which is a contradiction. Then set $L = L' + r/\pi$ the norm comes from the case 2 and $t = 1 - \gamma(r)$; for equivalent norms the unit ball is equivalent to L or L'. □

3.3 Ends

Definition 3.7. *Let $([L_0], [L_1], \ldots)$ be a infinite non-Backtracking sequence of adjacent vertices in which two sequences are equivalent if they differ by finite initial sequence of vertices.*

An equivalence class of such sequences is called an **end** of the tree. The set of ends is denoted $Ends(X)$ an represent the set of points at infinity for the tree.
Given an end $e = \langle [L_0], [L_1], \ldots \rangle$ we can construct a representing sequence of lattices for the path

$$L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \ldots$$

with the property that $L_i/L_{i+1} \cong \mathcal{O}_K/\pi\mathcal{O}_K$

Lemma 3.8. *The intersection of the lattices is a one dimensional subspace of V^* spanned by a linear form l.*

Proof. Since the sequence has no backtracking using the same argument that we use in the proof of the proposition that X is a tree we can prove that L_0/L_i is a cyclic $\mathcal{O}_K-$module of lenght i $\forall i \geq 1$ and the same is true for $L_i/\pi^i L_0$ so we may choose $l_i \in L_0/\pi L_0$ so that

$$L_i = \mathcal{O}_K l_i + \pi^i L_0$$

similarly

$$L_{i+1} = \mathcal{O}_K l_{i+1} + \pi^{i+1} L_0$$

Because $L_{i+1} \subsetneq L_i$ we must have

$$l_{i+1} = a l_i \mathrm{mod} \pi^i L_0$$

with $a \in \mathcal{O}_K$ because $l_i, l_{i+1} \in L_0 \setminus \pi L_0$, so we may a choose a coherent sequence l_i converging to l, which is non zero and $l \in \pi L_i$ and this intersection is one dimensional. The kernel of l is a point of $\mathbb{P}^1$ denoted by $N(e)$. □

Lemma 3.9. *The map from $End(X) \longrightarrow \mathbb{P}^1$, $e \longmapsto N(e)$, is a bijection.*

Proof. Let $L_0 = e_0\mathcal{O}_K + e_1\mathcal{O}_K$. Given $[x : y]$ in $\mathbb{P}^1$ written with unimodular coordinates. Let $l = -ye_0 + xe_1 \in L_0$ the end

$$\left\langle L_0, l + \pi L_0, l + \pi^2 L_0, \ldots \right\rangle \longmapsto [x : y].$$

Conversely we showed above that if l is a generator for the intersection of the sequence of lattices L_i representing and end

$$\langle [L_0], [L_1], [L_2], \ldots \rangle$$

then we must have $L_i = \mathcal{O}_K l + \pi^i L_0$ and so the map is bijective. □

3.4 The redution map

Given $x \in \mathcal{X}(\mathbb{C}_p)$ represented by homogeneous coordinates $[a, b]$ we obtain a norm γ_x on V^* (defined up equivalence) by setting

$$\gamma_x(l) = \omega(l(a, b))$$

for a linear form in V^*. The map $\gamma : \mathcal{X} \longrightarrow X$, $x \longmapsto [\gamma_x]$ is called **the reduction map.**

Lemma 3.10. *The reduction map is $G-$equivariant, so $g(\gamma_x)(l) = \mathcal{X} g_x(l)$.*
Let $L_0 = \langle e_0, e_1 \rangle$, $L_1 = \langle e_0, \pi e_1 \rangle$ then

$$\gamma^{-1}([L_0]) = \{[x, 1] \text{ s.t. } x \in \mathbb{C}_p \text{ and } \omega(x - t) = 0 \ \forall t \in \mathcal{O}_K\}$$

and if e is the open edge determined by $[L_0]$ and $[L_1]$ is the admissible annulus

$$\gamma^{-1}(e) = \{[x, 1] \text{ s.t. } x \in \mathbb{C}_{\mathbb{P}} \text{ and } 1 > \omega(x) > 0\}$$

References

[1] Dasgupta, S. and Teitelbaum J. The p-adic upper half plane. *Lectures Arizona Winter school (2007)*.

[2] Fresnel, J. and Van der Put, M. (2003)*Rigid Analytic Geometry and Its Applications* **218**.