Math Prep Camp: Sets and Functions

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Exercise sheet 1

1. *Membership (1)*
   Fill in correctly either $\in$ or $\notin$:
   (a) -3 _ $\mathbb{N}$, -3 _ $\mathbb{Z}$, -3 _ $\mathbb{Q}$, -3 _ $\mathbb{R}$
   (b) $\frac{1}{2}$ _ $\mathbb{N}$, $\frac{1}{2}$ _ $\mathbb{Z}$, $\frac{1}{2}$ _ $\mathbb{Q}$, $\frac{1}{2}$ _ $\mathbb{R}$

2. *Sets via enumeration.*
   (a) Define via enumeration the set of even integers between 3 and 9.

   (b) Define via enumeration the set of all capital letters from A to G (in alphabetic order).

3. *Two important rules.* Which ones of the following sets are equal:
   $A := \{a, d, b, a, d\}, B := \{a, d, d, a\}, C := \{a, d, c, a\}, D := \{b, a, d, d\},$
   $E := \{a, b, c, d\}, F := \{d, a, a, d\}, G := \{d, a, b\}.$

4. *Sets by properties.*
   (a) Define the following set via a property: $\{2, 4, 8, 16, 32, 64, 128, ...\}$.

   (b) Define the following set via a property: $\{-2, -1, 0, 1, 2\}$.

   (c) Define the emptyset via a property.

5. *Subsets and equality of sets*
   If possible, fill in correctly one of the symbols: $=$, $\subseteq$, $\supseteq$.
   (a) $\mathbb{N}$ ______ $\mathbb{Z}$
   (b) $\{1,5,3,4\}$ ______ $\{4,1,5\}$
   (c) $\{1,5,3\}$ ______ $\{4,1,5\}$
   (d) $\emptyset$ ______ $\{0\}$
   (e) $\emptyset$ ______ $\{n \in \mathbb{N} \mid n < 0\}$.

6. *Operations on sets*
   Let $A := \{a, d, g\}$, $B := \{e, b, f\}$, $C := \{a, b, c\}$. Compute:
   (a) $A \cup B$
   (b) $A \cup C$
   (c) $A \cup B \cup C$
   (d) $A \cap B$
   (e) $A \cap C$
   (f) $A \setminus C$
   (g) Which of the unions $A \cup B$, $A \cup C$, $B \cup C$ is disjoint?

7. *Cartesian product*
   Let $A := \{a, d, g\}$, $B := \{e, b, f\}$. Write by enumeration $A \times B$.

8. *Power sets (1)*
   List the elements of $\mathcal{P}(\{0, 1, 2\})$.

9. *Membership (2)*
Fill in correctly either $\in$ or $\notin$:

(a) Let $E = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$:
$3 \_ E$, $2 \_ E$, $-3 \_ E$, $\frac{1}{3} \_ E$

(b) $\emptyset \_ \{\emptyset, \{\emptyset\}\}$.

(c) $4 \_ \{\{4\}\}$.

10. *Equality of sets*
Which of the following sets are equal?
$A := \{1, 5, 9 - 2, 3\}$,
$B := \{x \in \mathbb{N} \mid 1 \leq x < 7 \wedge (2 \text{ does not divide } x)\}$,
$C := (\{x \in \mathbb{R} \mid x \geq 1\} \setminus \{2, 4, 6\}) \cap \{x \in \mathbb{R} \mid x < 8\}$,
$D := \{x \in \mathbb{N} \mid 1 \leq x \leq 8 \wedge (2 \text{ does not divide } x)\}$,
$E := \{x \in \mathbb{R} \mid 1 \leq x \leq 8 \wedge x \notin \{n \in \mathbb{N} \mid n \text{ is even }\}\}$.

11. *Subsets and equality of sets (2)*
Consider the following sets:

$$A = \{1, 2, 5\},\ B = \{\{1, 2\}, 5\},\ C = \{\{1, 2, 5\}\},\ D = \{\emptyset, 1, 2, 5\},$$
$$E = \{5, 1, 2\},\ F = \{\{1, 2\}, \{5\}\},\ G = \{\{1, 2\}, \{5\}, 5\},\ H = \{5, \{1\}, \{2\}\}.$$

(a) Which sets are related by equality and which by inclusion?

(b) Compute the cardinality of each of these sets.

(c) Determine $A \cap B$, $G \cup H$ and $E \setminus G$.

12. *Complements*
Consider the following four subsets of $\mathbb{N}$:

$$I = \{1, 2, 3, 4, 5, 6, 7\},\ J = \{1, 3, 5, 7\},\ K = \{2, 4, 6\}.$$

(a) Determine $I \setminus J$ and $I \setminus K$ (i.e. the complements of J and K in I).

(b) The symmetric difference of two sets A and B, denoted by $A \Delta B$, is the set of elements that are either in A or in B, but not in $A \cap B$. Determine $I \Delta J$ and $J \Delta K$.

13. *Intersection, union, complement, etc.*
Let

$$A = \{n \in \mathbb{N} \mid n \text{ is divisible by } 2\} \text{ and } B = \{n \in \mathbb{N} \mid n \text{ is divisible by } 5\}.$$

(a) Describe the intersection $A \cap B$.

(b) Describe the union $A \cup B$.

(c) Describe the complement $B \setminus A$.

(d) Describe the complement $A \setminus B$.

(e) Give the cardinality of $[12, 27] \cap A$ and of $[12, 27] \cap B$.

14. *Power sets (2)*
List the elements of:

(a) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.

(b) $\mathcal{P}(\mathcal{P}(\{1, 2\}))$.

15. *Proofs of equivalences involving sets.*
Let A and B be sets. Prove:

(a) $A \subseteq B \iff A = A \cap B \iff B = A \cup B$;

(b) $A \cap B = \emptyset \iff A \setminus B = A$.

16. *More proofs of equivalences involving sets.*
Let E be a set and A, B subsets of E. Prove:

(a) $A \cap B = \emptyset \iff B \subseteq E \setminus A \iff A \subseteq E \setminus B$;

(b) $A \cup B = E \iff E \setminus A \subseteq B \iff E \setminus B \subseteq A$.

17. Let S be a set and let $A \subseteq S$ and $B \subseteq S$ be subsets. Consider the following statements:

(1) $A \sqcup (S \setminus A) = S$

(2) $S \setminus (S \setminus A) = A$

(3) $A \subseteq B \Leftrightarrow (S \setminus B) \subseteq (S \setminus A)$

(4) $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$

(5) $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

Let $x \in S$ and let a be the assertion $x \in A$ and b be the assertion $x \in B$.

For each of the above statements, translate it into a statement from logic involving the assertions a and b, which you should recognise as a well-known rule from logic.